

Non-Relativistic Bose-Einstein Condensates, Kaon droplets, and Q-Balls

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We note the similarity between BEC (Bose-Einstein Condensates) formed of atoms between which we have long-range attraction (and shorter-range repulsions) and the field theoretic "Q balls". This allows us in particular to address the stability of various putative particle physics Q balls made of non-relativistic bosons (K^0 's, B^0 's, and D^0 's) using variational methods of many-body NRS (Non-Relativistic Schrödinger) equation.

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Introduction

Phase transitions occur when a change of coupling or temperature makes the (free) energy of a new phase lower than that of the preexisting phase. In the field theoretic formulation, this is manifest when the minimum of the effective potential $U(\phi)$, with the field ϕ representing some order parameter, is shifted from $\phi = 0$ to nonzero ϕ value or to a degenerate manifold of such ϕ values. In a simple mechanical analog the system represented by the single coordinate ϕ , "rolls over" to the new stable minimum. Some time ago, Sidney Coleman introduced[1] "Q balls": classical field configurations stabilized by the global conserved charge (s) (Q) they carry. These correspond to configurations of, say, a charged field which are constant over a large region of space but vary in time: $[\phi \exp(-i\omega_0 t)]$ and $[(\phi)^+ \exp(i\omega_0 t)]$ —corresponding to a constant charge density $j^0 = \rho = (\frac{d}{dt}\phi^+ \cdot \phi - \phi^+ \frac{d}{dt}\phi)/2i$. The simple mechanical analog here is a system rotating with uniform angular velocity in the $\phi_1 - \phi_2$ plane (with $\phi = \phi_1 + i\phi_2$). The "centrifugal" force generated can then, under certain conditions specified below, make the representing "particle" come to equilibrium at a ϕ_0 which is no longer a minimum of $U(\phi)$. Coleman's suggestion of using the conserved baryon number as the global charge of the Q balls has been followed up in supersymmetric models[2]. Motivated by advances in BECs (Bose-Einstein condensates), we investigate in this paper their relation to Q balls and use variational NRS (non-relativistic Schrödinger) equation methods to prove the stability against strong interaction decays of Strangeness, Charm, and Beauty balls.

Field theoretic Q balls are more general than the non-relativistic limit on which we focus here. Thus[2] the above-mentioned baryonic Q balls, made of squark condensates, are stable against "weak interaction like" decays of the heavy individual squarks by having the Q balls very tightly bound with masses proportional to a fractional power of the total baryon number $N^{(1-\epsilon)}$ rather

than N^1 as expected for non-relativistic weakly bound matter.

Still, the equivalence—in some limits—of the field-theoretic and many-body descriptions of the same "condensations" is interesting and helpful.

There are *two* distinct, though interrelated, types of condensation. BECs obtain when a large number, N , of (bosonic) atoms are trapped within a radius R and then cooled down to nano-Kelvin temperatures. The BE condensation is in *momentum* space: a finite fraction of the atoms are in the lowest mode of the trap corresponding to $p = 0$. This manifest when the trap is suddenly removed by having these $p = v = 0$ atoms hardly move.[3] Boson-boson interactions modify the BEC but are *not* responsible for BEC phenomenon in the first place. BEC is best understood when the atoms are non-interacting[4]. On the other hand the *coordinate* space "condensation" of bosons into Coleman's "Q balls" is due to attraction between the bosons. Thus in the field theoretical formulation Coleman has proved that stable Q balls exist if and only if the "potential" $U(\phi)$ in the effective low-energy Lagrangian for the system satisfies:

$$(i) \quad \frac{U(\phi)}{\phi^2} \text{ has a minimum lower than } \frac{\mu^2}{2} \quad (1)$$

with μ the mass in the "free" part of $U(\phi)$: $\frac{\mu^2}{2}\phi^2$.

The size R of the spherical Q ball and its density $n = (3Q)/(4\pi R^3)$ are fixed by the overall Q and ϕ_0 —the field for which the above minimum is achieved.

Condition (i) implies over all attractive interactions between the ϕ bosons, say, the kaons in strangeness balls, at an appropriate density. In particular, (i) holds if the coefficient of the lowest $(\phi)^4$ term in $U(\phi)$ is negative:

$$(ii) \quad U(\phi) = \frac{\mu^2}{2}\phi^2 - \lambda\phi^4 + \text{higher order terms} \quad (2)$$

corresponding to attractive S-wave scattering length ($\lambda > 0$ is implicit). Note that $U(\phi)$ is an *effective* Lagrangian,

which is *not* used in loops inside Feynman diagrams. Therefore $U(\phi)$ can have (and indeed has) higher-order *positive* non-renormalizable $(\phi^+ \phi)^n$ terms—ensuring a finite ϕ_0 and a spectrum which is bounded from below.

While (ii) \rightarrow (i), namely, an attractive S-wave scattering length (negative ϕ^4 coefficient), implies stable Q balls, (ii) is *not* required. Thus, a nontrivial minimum, ϕ_0 , of $U(\phi)/(\phi)^2$ can obtain with a negative $(\phi^+ \phi)^3$ term overcoming at some ϕ the positive $\frac{\mu^2}{2}(\phi)^2 + \lambda(\phi)^4$. Still this can be problematic: the minimum of $U(\phi)/(\phi)^2$ may now be at a large ϕ_0 , say, $\phi_0 \gg \mu$ where the NR many-body approach that we want to compare with next, fails.

Note that Condition (ii) alone, without knowledge of the higher-order terms, does *not* fix the size (R) or the density (n) of the Q balls: with only the $[-\lambda\phi^4]$ term present both ϕ_0 and n are infinite!

Forming a Spatial Droplet of Non-relativistic Bosons: The Many-Body Approach

Let N non-relativistic identical bosons of mass m interact via potentials $V(|r_i - r_j|)$. The basic question we address is: “For which potentials the NR bosons coalesce into “Q balls” with nonzero density when $N \rightarrow \infty$?”

For finite range potentials the N-body Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$ with

$$H = \sum_{(i)} \frac{p_i^2}{2m} + \sum_{(i>j)} [V(|r_i - r_j|)] \quad (3)$$

is trivially solved by Ψ which is a product of single particle wave functions, each of which is (approaching) a constant. The spread-out particles have vanishing kinetic energies and vanishing mutual interactions and thus $E = 0$. For many potentials a lower (negative) energy state exists with the N bosons in a sphere of radius R . Using the basic variational principle we derive next several *sufficient* conditions for that to happen. In Secs. III, IV below we argue that some of these sufficient conditions are met in the case of many K^0 's, D^0 's and B^0 's which therefore will make droplets or Strangeness, Charm and Beauty balls which are strong interaction stable.

Let us first use a simple trial wave function with all bosons in the same state—the ground state of a large radius R spherical cavity:

$$\Psi_t = \prod_i (\psi^0(r_i)). \quad (4)$$

The expectation values of the kinetic and potential energies appearing in $\langle \Psi_t | H | \Psi_t \rangle = \langle K \rangle + \langle V \rangle$ are:

$$\langle K \rangle \simeq N\hbar^2/(2mR^2) \quad (a) \quad (5)$$

and, if $R \gg r_0 =$ the range of the potential, $\langle V \rangle \simeq [Nnv] \sim [N^2/(4\pi R^3/3)]v$, with

$$v = 4\pi \int V(r) r^2 dr \quad (b) \quad (5)$$

and $n = N/(\text{volume})$ the particle number density. A possible relation to the previous section stems from the fact that, up to kinematic factors, v is the Born approximation for the S-wave boson-boson scattering length. For dilute systems with inter-particle separation which far exceed the range of the potential:

$$d = n^{-(1/3)} \gg r_0, \quad (6)$$

only the integrated potential v effects the threshold scattering. The threshold scattering amplitude is therefore reproduced also by a “pseudo-potential” of a local delta function form: $v\delta^3(r)$. Such a potential roughly corresponds to the negative local $-\lambda(\phi^4)$ term—which in the field theoretic formulation suffices to ensure stable Q balls. To complete the analogy with the NRS case we show that if $v < 0$, then also the expectation value of the energy $\langle H \rangle < 0$. This readily follows from the different scalings of the (expectation values of) the positive kinetic and negative potential energy N , the number of bosons in the system: $\langle K \rangle \sim N^{(1/3)} \ll |\langle V \rangle| \sim N$ for $N \rightarrow \infty$. By the variational principle the energy of the true N-body ground state is lower than $\langle H \rangle$ and also negative and a many-body bound droplet or Q ball stable state indeed exists if:

$$(ii') : v = 4\pi \int V(r) r^2 dr < 0 \rightarrow \text{a “droplet” state exists.} \quad (7)$$

Like (ii), (ii') does *not* fix the actual size/density of the “droplet”. Still the two conditions are *not* equivalent.

The quantity v , depends only on the potential $V(r)$ and not the mass m . It is (proportional to) the actual S-wave scattering length (or to the coefficient of the ϕ^4 term in $U(\phi)$ *only* in the Born (or dilute system) approximation.

Scattering theory[5] implies that the S-wave scattering length is attractive if and only if the NRS (non-relativistic Schrödinger) two-body system has bound states. Thus the NRS equivalent of the field theoretic condition (ii) is having a two-body bound state of the NRS equation. Indeed as we directly show below having two-boson bound states guarantees N bosons bound state. Note, however, the independence of the two NRS criteria: The criterion (ii') ensuring $N \rightarrow \infty$ NRS “droplets” does *not* ensure an S-wave two-body bound state. In three dimensions the latter requires not only an “attractive” potential, but also sufficiently strong attraction.

Various criteria for $V(r)$ to have bound states in a NR two-body system with reduced mass m exist.[6] Yet there is no general *if and only if* criterion, short of solving the Schrödinger equation. Finding if a bound state of $N \rightarrow \infty$ bosons exists need not be easier. Indeed the field theoretic Criterion (i) requires the full effective potential $U(\phi)$. The coefficients in the power series for the latter are the threshold scattering amplitudes for any number

of particles and cannot be found short of solving exactly the field theory.

“Strangeness-Beauty Balls” and the Non-Relativistic Schrödinger Equation

For many atomic and other systems, the many-body NRS treatment preceded field theoretic approaches and Laughlin’s celebrated variational wave function for the fractional quantum Hall effect is a prime example of this. In recent years much effort has been devoted to applying field theory in general, and effective field theories and effective Lagrangians in particular, to such problems and this has been also the case for the Q balls.[7]

Chiral Lagrangians were used to check if Coleman’s Criterion (i) holds for the K^0 system. These Lagrangians involve higher derivative terms and are fixed by an overall fit to data yet they do not offer much intuitive understanding. Here we follow the *reverse* program of “demystifying” Q balls and trying to explain them—at least in the NRS regime—as simple spatial condensations of many bosons with appropriate potentials. Strangeness-balls, and even more so, Charm/Beauty-balls with densities $n < m_K^{-3}$ are NR. In all cases we expect to have essentially the same boson-boson potentials as the later are controlled by the common light d quark. Hence, using $K^0 - K^0$ potentials, $V_K(r)$, etc., to find if “ K^0 , etc., Droplets” form—which we do next—is justified.

A first key observation is that the $K - K$ potential, as that between *any* two identical, neutral, pseudo-scalars, is *attractive* at “large” \sim Fermi distance. To show this we write $V(r)$ as a superposition (integral) of Yukawa potentials due to all exchanges[8]:

$$V(r) \simeq - \int d\mu \sigma(\mu^2) \frac{e^{-\mu r}}{r} \quad (8)$$

Parity conservation forbids a $KK\pi$ vertex and the lightest exchanged system controlling $V(r)$ at $r \rightarrow \infty$ is that of two pions. Further, the $\ell = 0$ component dominates at the $\pi\pi$ threshold:

$$\sigma(\mu^2) \text{ near } \mu = 2m(\pi) \sim |f(\ell=0)(K + \bar{K} \rightarrow \pi\pi)|^2 \quad (9)$$

With $f(\ell=0)$, the $\pi\pi \leftrightarrow K\bar{K}$ S-wave amplitude. This expression is clearly positive and recalling the minus sign in the definition of $V(r)$ we find that the longest range two-pion exchange potential is indeed attractive. This can be also directly shown to be the case for the exchange of a 0^{++} state which can be an S-wave resonance in the $\pi\pi$ S-wave system. The above reasoning is similar to that used in [8] to derive the well-known attractive Casimir Polder (i.e retarded van der Waals) and the regular van der Waals two photon-exchange potential between two identical neutral atoms.

At short distances of the order of the size of the $s\bar{d}$ composite state, which *is* the kaon in quark models/QCD, the

KK potential becomes repulsive. As in atomic physics this is due to the Pauli principle—operating here between the identical \bar{d} or s quarks in the two K^0 ’s. The repulsion can also be viewed as being due to the exchange of the ω (ρ) vector mesons: The two K^0 ’s have the same hypercharge (isospin) to which ω (ρ) couple[9]; no light 0^{++} , scalar “ σ ” meson has been established. Yet, the exchange of such an entity with appropriate mass and coupling to nucleons $g(\sigma; NN)$ (which may represent the box diagram with two pions exchanged and intermediate N and $\Delta(1230)$ states) could, along with ω exchange, dominate nuclear binding[10].

Parameterizing,

$$V_K(r) = -g(\sigma)^2 \frac{e^{-(m_\sigma r)}}{r} + g(V)^2 \frac{e^{-m_V r}}{r} \quad (10)$$

with $V = \omega$ or ρ at a common mass $m(V)$ and $g(V)^2$ the sum of the (squared) ρ and ω couplings to kaons we find:

$$v_K = -\frac{g(\sigma)^2}{m(\sigma)^2} + \frac{(g(\rho)^2 + g(\omega)^2)}{m(V)^2} > 0 \quad (11)$$

To evaluate (11) we take $g(\rho, KK) = g(\omega, KK) \sim (1/3)g(\omega, NN)$ and $g(\sigma, KK) \sim (1/3)g(\sigma, NN)$, as suggested by counting the numbers of non-strange quarks. Using the values of $g(\omega, NN)^2/m_V^2$ and $g(\sigma, NN)^2/m(\sigma)^2$ suggested by fitting nuclear matter in Eq. (14.27) in [10] we find that $v_K \sim (-2.35 + 3.45)/m_N^2 > 0$, so that Criterion (ii’) is *not* satisfied. The box diagrams with K^* intermediates for KK suggests a $g_{\sigma, KK}^2$ which is somewhat bigger than the previous $[(1/9)g(\sigma, NN)^2]$ estimate. Also cutting off $V_K(r)$ of Eq.(10) in evaluating v_K at, say, 0.3 Fermi—suggested by the fact that the K^0 is composite at such a scale and can no longer be treated as a point source of the σ, ω fields—further reduces the repulsive relative to the attractive contribution and a negative v_K is not excluded.

Still $v_K > 0$ is likely and we face the question: “Does a positive v_K exclude stable K^0 droplets?” This is the case if we insist on N-body wave functions which are products of N identical *one*-particle wave functions $\psi(r_i)$. However, including the (Jastrow) product of $N(N-1)/2$ *two*-particle functions

$$\Psi_{(trial)} = \prod_i (\psi(r_i)) \prod_{(i>j)} (f(|r_i - r_j|)) \quad (12)$$

we can have an N-body bound state even if $v > 0$.

To illustrate this we fix the potential $V_K(r)$ (and v_K) and increase the mass. (Taking $m = m_B \sim 11m_K$ corresponds, in the approximation of universal potentials between pairs of heavy mesons—and treating s as a heavy quark, to discussing “Beauty” (rather than Strangeness) balls. While $v_K \sim v_B \sim v$ the system can now have even two-body bound states.

In the $m \rightarrow \infty$ limit these are localized around the minimum of $V(r)$ at $r = r_0$, forming a “Vibrational Band”

with spacings $\sim ([V''(r)|r=r_0]/m)^{(1/2)}$. Once the potential $V(r)$ has two-body bound states, N-body bound states are guaranteed. This is verified by using in Eq. (12) $f(r) = \psi_0(r)$ with $r = |r_i - r_j|$ satisfying:

$$\left(-\frac{\hbar^2}{m} \frac{d^2}{dr^2} + V(r)\right)\psi_0(r) = E_0\psi_0(r) \quad (13)$$

with $E_0 < 0$ the negative energy of the two-body bound state, and operating on the above trial function with the N-body Hamiltonian of Eq. (3) above. (Here, the reduced pair mass $m^* = m/2$ replaces the single particle mass in the Schrödinger equation.) Even when the potential is too weak relative to the kinetic term to have a two-body bound state, using Eq. (12) with $f(r)$ of the above general form, namely, peaking at the minimum of the potential and being small at small r 's where $V(r)$ is repulsive, lowers $\langle H \rangle$ relative to its value for the product of one-particle functions. As we show in some detail in the next section, this can yield the desired N-body bound state even when Condition (ii') fails, and also there are no two-body bound states. In this case we have a fully symmetric N-boson bound state with the $f(|r_i - r_j|)$ factors peaking at $r = r_0$, and determining the density of the N-boson droplet to be:

$$n \sim [(4\pi)/3]r_0^3(-1) \quad (a) \quad (14)$$

or, equivalently, the radius of the droplet

$$R \sim N^{(1/3)}r_0 \quad (b) \quad (14)$$

This is reminiscent of $U(\phi)/(\phi)^2$ and its nontrivial minimum at ϕ_0 —fixing the radius and density of the Q balls in the field theoretic formulation. There the nontrivial ϕ_0 obtains via the interplay between a negative ϕ^4 term and positive higher-order terms. In the present NRS case the minimum at r_0 reflects attraction (repulsion) at long (short) ranges. The higher $(\phi)^n$ terms are prominent at large densities—just like the strong short-range repulsions in NRS.

Still, a $U(\phi) \leftrightarrow V(1/r)$ analogy is rather limited: The *effective* potential U derives from the fundamental Lagrangian of the field theory, say, QCD for the above cases, whereas the potential $V(r)$ is the primary entity in the NRS approach. A closer analog of $U(\phi)$ is the derived quantity $[E/N](n)$ —the energy per particle for a given density [11] in NRS. $E = E[N; R]$ is the ground state energy of the Hamiltonian in Eq. (3) of $N = n[(4\pi)/3] \cdot R^3$ bosons uniformly distributed (after averaging over correlations) in a sphere of radius R . When N and R tend to ∞ keeping n fixed, a stable droplet of density n_0 obtains that if and only if the minimum of $[E/N](n)$ is at $0 < n_0 < \infty$ and is negative [12].

Binding and BECs in the Presence of Strong, Short-Range Repulsions

Short-range repulsions do not hinder BEC for dilute atomic systems in traps.

Let N-bosons be in the trap and add one more. To see the issue most clearly, assume first that the N-bosons are “frozen” at specific locations r_i^0 inside the trap. The added $k \ll N$ bosons will be in the ground state of the total potential:

$$V = V_{(trap)}(r) + \sum_{(i=1, \dots, N)} V(|r_i^0 - r|) \quad (15)$$

—provided that the potential 15 can bind a particle of mass m (which is clearly the case for an attractive $V_{(trap)}$ and sufficiently dilute atoms).

Conceivably such a setup, of interest in its own right, can be experimentally realized. Let the trapped N atoms form a 3-D lattice generated by a standing wave pattern of three lasers. Let the added atoms be of a *different* species which interact with the first N atoms via the potential V in Eq. (15) above. In particular, we need to choose a species which is almost unaffected by the laser fields.

“Freezing” N out of $N + 1$ is artificial and adding one extra boson causes each of the previous N -bosons to adjust by order $1/N$, modifying the binding energy by $O(1)$. A key point is that the adjustments are likely to *lower* the energy and neglecting those is appropriate if we only want to verify that the extra particle binds.

As the system becomes denser, stronger two-boson correlations and higher momentum components build up. The completely symmetric N-body state, while no longer factoring into independent N single-particle wave functions, still exhibits coherence features unique to BECs.

Our main interest however is not this, but rather the “droplet formation” problem posed in the previous section. To address this problem we use Eq. (15) without $V_{(trap)}$ and two-body potentials which are attractive (repulsive) at long (short) distances (Fig 1.)

A likely configuration the N particles r_i^0 is at the vertices of a simple cubic lattice of lattice constant $d = n^{(-1/3)}$. A simplified version of the problem which serves as a criterion for formation of a droplet of density $n = d^{-3}$ is:

(iii) “Does a particle of mass m bind to a cubic lattice with lattice constant d and common potentials $V(|r - r_i|)$ centered at all lattice points r_i ?”

—a problem which may also be useful in discussing trapping of light in some “dielectric lattices” via the Helmholtz equation.

It is difficult to address it in the most general case, yet the following suggests that bindings are likely even when a single potential $V(r)$ fails to bind and/or to satisfy $v <$

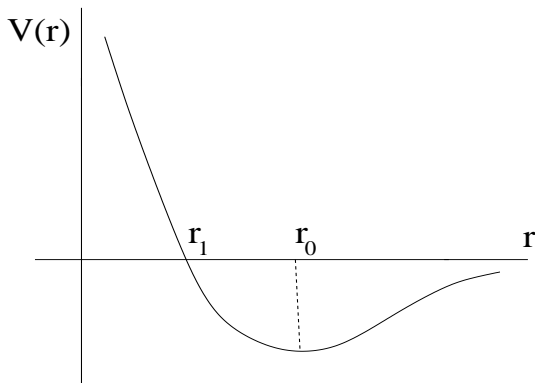


FIG. 1: The KK, DD or BB NRS potential of interest. It is repulsive at short distances $r < r_1$ and attractive for $r > r_1$ having an asymptotic $\frac{e^{-2m\pi r}}{r^3}$ behavior due to two-pion exchange (qualitatively similar to a Lennard-Jones potential). The minimum of the potential is indicated by the dotted line is at r_0 .

0. Let r_0, r_1 with $r_0 > r_1$ be the points indicated in Fig. 1 where the common radial potential $V(r)$ has its minimum and where it changes sign, being repulsive (attractive) for $r < r_1$ ($r > r_1$). Consider a unit lattice cell of side of length d with eight potential centers at its corners. We focus first on the “dilute” case with $d \gg 2r_0 > 2r_1$. Apart from the eight spherical octants of radius r_1 at the corners, V of Eq. (15) above is attractive at all the remaining part of the unit cell. The longer-range tails of the other potential centers make for an attractive, negative contribution in the form of a “Madelung sum” at any, (say, corner) point. This increases the volume of the connected region within the unit cell where V (sum) < 0 beyond the minimal value:

$$d^3 - \frac{4\pi}{3}r_1^3 \quad [\sim .93 d^3 \text{ if } d > 4r_1 !]. \quad (16)$$

Thus we find that the particle is inside a (very loose!) cage of size $\sim d^3$ “cornered in” by the repulsive potentials centered at the eight corners of the unit cell of the cubic lattice—as in a three-dimensional analog of a carton egg holder. Actually, in the dilute case considered here, the particle can “roam” over all the lattice, going into neighboring cells via the large circular openings of radius $d/2 - r_1$ between neighboring cells, lowering the energy. To show this more clearly we simplify the problem by: (a) replacing the repulsive potentials within spheres of radius r_1 centered at the vertices of the lattice by infinite, positive, “square-well” potentials within the circumscribing cubes of size $2r_1$ centered—just like the above spheres—at the corners of the lattice cell, and (b) replacing the attractive potential within the remaining region by their volume average,

$$-u_0 = v^-/d^3 \quad (17)$$

with v^- , the integral over the attractive part of the po-

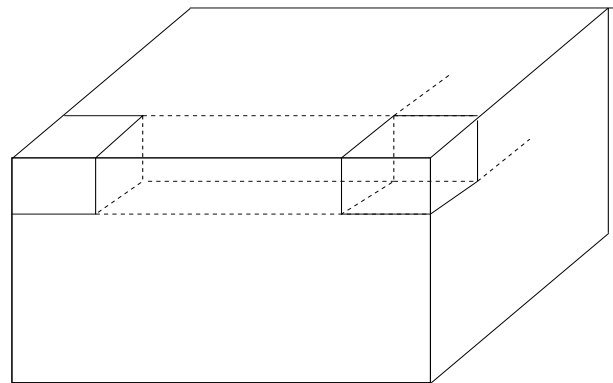


FIG. 2: A schematic picture of the attractive and repulsive regions (after step b has been implemented) inside the unit cell—the overall cube of side d . The repulsive regions (with $V = +\infty$) are the eight smaller cubes of side r_1 at the eight corners of the unit cell—the big overall cube. Three of these are shown as the full line small cubes in the figure. The potential is attractive (negative) outside these eight cubes with an averaged constant value. Using additional dashed lines we also illustrate two out of the twelve rectangular parallelepiped (RP) along the edges of the cube, between a pair of adjacent repulsive boxes. These can be viewed as “wave guides” for the particle in the potential where we have vanishing boundary conditions on the small (r_1^2 in area) sides of the RP and the particle is propagating in directions perpendicular to the long ($d - 2r_1$ in length) sides of the RP e.g., in the top-front RP the wave guides in the up-down and forward-backward directions, and in the right-most RP illustrated in the up-down and the left-right directions.

tential. (See Fig. 2.)

Step (a) increases the volume of the region where the potential is repulsive by $6/\pi$ and the value of the potential which vanishes at its boundary to ∞ . This clearly increases the energy of the particle in the lattice. It is less obvious but still true that performing step (b), namely, replacing the attractive part of the potential by its average value, also increases the energy of the particle in the lattice. To see this, note that the true ground state wave function for the original potential tends to concentrate away from the repulsive regions and thus can sense better the attractive potential prevailing around the center of the above “cage”, i.e., unit lattice cell, and this attractive potential is *stronger* before the averaging is performed. Hence, if after performing steps (a) and (b) above the particle still binds to the lattice, namely, ($E_{\text{ground}} < 0$), then even deeper bound states are likely for the original problem.

To test for a bound state we estimate the expectation value of the Hamiltonian $\langle H \rangle = \langle V \rangle + \langle T \rangle$ in the state $|\psi_{\text{ground}}\rangle$. Since the (normalized) wave function is nonzero only outside the repulsive regions, the expectation value of the constant attractive potential is:

$$\langle V \rangle = -u_0 = v^-/d^3. \quad (18)$$

The kinetic energy stems from the boundary conditions: $\psi = 0$ at the boundaries of the repulsive cubes. The twelve regions, of volume $r_1^2(d-2r_1)$ each, inside the d^3 unit cell between pairs of repulsive corner cubes can be viewed as short section parts of “wave guides”. The kinetic energy can be roughly approximated as that of the lowest mode of the “wave guides”: $T = \hbar^2/[2m(d-2r_1)^2]$ and its expectation $\langle T \rangle$ is weighted by the fraction of the unit cell, $f = 12r_1^2(d-2r_1)/d^3$, occupied by the “wave guide” sections, namely

$$\langle T \rangle = fT = \frac{\hbar^2}{m} \frac{6r_1^2}{d^3(d-2r_1)}. \quad (19)$$

The condition for binding, $\langle T \rangle + \langle V \rangle < 0$ then becomes

$$|v^-| > \frac{\hbar^2}{m} \frac{6r_1^2}{d-2r_1}. \quad (20)$$

Since v^- is independent of d , the last condition can clearly be satisfied for $d \gg r_1$.

As we gradually decrease d (relative to the distance scales r_0 and r_1 of the potential $V(r)$), the average value of the attractive part, namely, $|u_0|$ of Eq. (18) above increases, thereby enhancing the binding.

However, once $d \sim r_1$ the “cages” trapping the particle within each unit cell become tighter and the particle can only tunnel between the different cells. The energy then rises and the bound state disappears.

Finding the optimal d (or density of the droplet n) and the corresponding binding: $[E/N](n)$ at this number density requires detailed calculations beyond what we have attempted here[13].

Showing that one extra particle can be bound in a periodic box of size L where the previous N are located at prescribed positions r_i^0 (as we did above when the r_i^0 were the nodes of a regular simple cubic crystal) is only *one* step towards proving the existence of large Kaon droplets with the specific KK potential above. Indeed:

i) While we optimize the inter-particle separation $d \sim N^{(-1/3)} \cdot L$ to minimize ϵ_N , the energy of the $N+1^{th}$ particle added to the lattice, we should verify that the same d also allows each of the original N particles to bind around the empty lattice site that it occupies.

ii) We should show that the particular frozen r_i^0 arrangement at *all* lattice sites within the periodic L^3 box represents the “worst case” for binding the $N+1^{th}$ particle and any rearrangement of the N bosons inside the box allows stronger binding of the $N+1^{th}$ boson.

iii) Using i) and ii) above we then show that enlarging the system from N to $N+1$ bosons *and* simultaneously letting the cube size adjust to the new optimal length, $L(N) \rightarrow L(N+1)$, the energy is lowered by *more* than the above binding:

$$E[(N+1), L(N+1)] < E[N, L(N)] + \epsilon_N \quad (21)$$

This last step is readily achieved by employing

$$\begin{aligned} E[(N+1), L(N+1)] = & \int \prod d^3 r_i^0 \int d^3 r_{N+1} \Psi(r_{N+1}; r_i^0) \\ & \times [(\sum_{i=1, \dots, N} -\frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i>j=1 \dots N} V(r_i^0, r_j^0)) \\ & -\frac{\hbar^2}{2m} \nabla_{(N+1)}^2 + \sum_{i=1 \dots N} V(r_i^0, r_{(N+1)})] \\ & \Psi(r_{N+1}; r_i^0). \end{aligned} \quad (22)$$

We use the 0 suffixes on the first N coordinates to emphasize that in evaluating the expectation of the kinetic energy and N interactions of the $N+1^{th}$ particle by doing the innermost $r_{(N+1)}$ integration, these first N coordinates are “frozen”. This integral then yields $\epsilon[r_1^0, \dots, r_N^0]$, the binding energy of the extra particle to the first frozen N which is then further averaged over all r_i^0 using the normalized measure provided by the density function of the first N particles. If (ii) holds this yields a value smaller than ϵ_N above corresponding to the special case of a perfect full lattice of size $L^3(N+1)$.

Next consider the first term in the last integral, namely, $H(N)$, the part of the Hamiltonian pertaining to the first N bosons. Since $H(N)$ does *not* depend on $r_{(N+1)}$, the Normalized integral over $H(N)$ simply yields $E[N, L(N+1)]$ which exceeds $E[N, L(N)]$. Adding the two terms we find at the desired inequality (21).

Summing these over $n < N$ we yield a lower bound on the binding of the droplet;

$$E_N < N\epsilon_N \quad (23)$$

For a potential which is an attractive constant apart from hard core cubes of size $2r_1$ around each boson, i.e., the case studied above this last inequality, is directly proven in one dimension in Appendix I. Unfortunately, this particular elegant method does not readily generalize to three dimensions.

Coming back to issues i) and ii), clearly filling up completely a simple cubic lattice with the previous N bosons—rather than leaving one “hole” i.e., a vacant lattice site where the extra particle nicely fits lowers the volume of the “cage” in which the N^{th} particle is free to roam. This, in turn, elevates its kinetic energy. The attractive potential energy is $\{8V[(\sqrt{3}/2)d] + 24V[(\sqrt{11}/2)d] + \dots\}$.

Compare this to the “optimal” regular arrangement with one vacancy in the regular cubic lattice. The free volume where no strong repulsion occurs is now \sim twice as large making roughly for $2^{-(2/3)} \sim .65$ times lower kinetic energy, whereas the attractive potential here is $6V[d] + 12V[d\sqrt{2}] + \dots$. For $d \sim r_2 \sim 2m\sigma^{-1}$ the latter is similar to the potential energy in the previous case. Hence we find that certain d values which allow binding of the $N+1^{th}$ particle to a perfect lattice make for even stronger binding inside the lattice.

Next let us consider a random rearrangement of the r_i^0 inside the L^3 cube with the same average number density Rd^{-3} . It will contain pairs, triplets etc., of particles which are nearer to each other than d and this will be compensated by having nearby “lacunas” of underdense points. The latter constitute ideal placements of the extra $N + 1^{th}$ particle: It will have more free space to move and at the same time will be more strongly attracted to the dense “clusters”. Note that in evaluating $\epsilon[r_1^0, r_2^0, \dots, r_N^0]$ the binding of the extra $N + 1^{th}$ particle to the frozen N particles at r_i^0 , we need *not* worry about the fact that the “crowding” of some of the frozen vertices raises *their* mutual interaction energies.

Some Concluding Remarks

In this paper we have shown a close correspondence between the field theoretic concept of Q balls and droplets of coherent NR bosonic matter and elaborated at some length on the criteria, in a NRS picture, for forming the such droplets.

While recently field theoretic/effective Lagrangian methods—such as those used by Coleman in predicting Q balls—are broadly applied to many-body physics, we find that the traditional variational NRS approaches can be used to prove the existence of some “particle physics” Q balls. Unfortunately these Q balls with $Q = \text{Strangeness, Charm and Beauty}$, while stable against decays via strong interactions do decay rather quickly via weak interactions. A single K^0 decays in $\sim 10^{-10}$ sec (and D^0, B^0 decay 100 times faster). A droplet of N non-relativistic, weakly bound neutral bosons will start disintegrating after times which are $1/N$ shorter than the decay time of a single boson. If the minimal number of K^0 's can be viewed as an $N \rightarrow \infty$ “droplet” is ~ 100 , we need to assemble within $\sim (5 \text{ Fermi})^3$ 100 slow K^0 's in a picosecond!, which seems impractical.

We would like to note, however, that the general features of the KK potential facilitating droplet formation, namely, attraction at “long” ~ 0 (Fermi) distances and repulsion at “short” $1/3$ Fermi distances, hold also for the K -Nucleon (and the $N - N!$) system. While there are both few body and “Droplets” of nucleons (a.k.a. “Stable Nuclei”) no $K - N$ bound states exist and the $K - N$ S-wave scattering length is known to be repulsive.

However K^0 droplets exist despite the absence of a KK bound state. Could a K^0 which is free to move in a pre-existing large nucleus (so long as it avoids getting too close to the nucleons) bind to the latter? This is clearly not evident since the nuclear density and size are fixed by nucleon-nucleon interaction and *not* by $K - N$ physics.

It is amusing to note, however, that the existence of such states would manifest experimentally as follows: Let a beam of slow K^+ charge exchange on a heavy nucleus with the resulting K^0 binding to the nucleus. The sub-

sequent $K^0 \rightarrow \pi^+\pi^-$ decay $\sim 10^{-10}$ sec later is likely to break the nucleus, yielding a spectacular “Star” which, in the absence of a K^0 -nucleus bound state, should not occur.

APPENDIX I

In the “frozen N ” variant, the $N + 1^{th}$ particle is restricted to the N intervals of size L'/N between pairs of existing particles (periodic boundary conditions avoid half size end intervals) with $L' = L - 2Nr_1$ the effective length allowed by the constraints. There is no tunneling between these N intervals and the energy is :

$$\epsilon_N = \hbar^2/[2m(L'/N)^2] \quad (24)$$

Note that $N\epsilon_N$ is the energy of *one* fictitious representative particle moving in an N dimensional cube of side L'/N . The no-tunneling rigidity which is an artifact of one dimension, reflects also in the full N -body problem of finding the ground state of the Hamiltonian $-\frac{\hbar^2}{2m} \sum \frac{\partial^2}{\partial x_i^2}$ for N particles in the $(0, L')$ interval via the impenetrability—or ordering condition—

$$0 < x_1 < x_2 < \dots < x_N < L'. \quad (25)$$

Equivalently we have one particle restricted to the above N dimensional parallelepiped (P_N) satisfying the free Schrödinger equation in N dimensions.

(The $N!$ parallelepipeds P_N 's obtained by permuting the x_i have the same volume $L'^N/N!$ and their union constitutes a cube of side L'). P_N contains the N -dimensional cube C_N of side L/N :

$$\begin{aligned} 0 < x_1 < L/N, \\ L/N < x_2 < 2L/N, \\ &\dots, \\ (k-1)L/N < x_k < kL/N, \\ &\dots, \\ (N-1)L/N < x_N < L, \\ \text{and } ||P_N|| > ||C_N||. \end{aligned} \quad (26)$$

The variational principle and the above containment relation imply a lower energy for the full problem.

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